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On the oscillatory instability of a differentially heated fluid loop

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A theoretical discussion is given of the motion of a fluid contained in a tube forming a closed loop that is heated from below and cooled from above. The fluid is assumed to have uniform temperature over each cross-section, and the heat transfer is assumed proportional to the difference between the local temperatures of the fluid and the tube. The latter temperature is prescribed. The system has one steady solution with warm fluid rising in one branch and cold fluid sinking in the other. This solution may, however, become unstable in an oscillatory manner. A weak instability takes the form of pulsations, the motion being always of one sign, while a strong instability takes the form of oscillations with zero mean motion. These oscillations are irregular and do not repeat themselves even over very long times.

These unstable motions are associated with thermal anomalies in the fluid that are advected materially around the loop. The anomalies amplify through the correlated variations in flow rate. A warm pocket of fluid creates maximum flow rate going through the upper part and minimum flow rate going through the lower part of the loop. Accordingly it passes quicker through the heat sink than through the heat source, and the latter becomes more effective. Similarly, the heat sink acts more effectively on a cold pocket of fluid.

The curve of neutral stability is worked out as a function of the two parameters of the problem, a non-dimensional gravity and a non-dimensional friction coefficient. The instability has also been studied by direct numerical time integration of the model equations.

It is suggested that the mechanism of instability found for this model operates also in more complicated systems, and can explain the pulsative type of motions observed recently in certain convection experiments.

1. Introduction

The problem of general thermal convection (fluid motion set up by heating and cooling processes) is so far only incompletely understood. Most attention has been paid to the case of Bénard convection (convection between two horizontal boundaries kept at different temperatures). The stability problem and the range of steady laminar convection is relatively well explored, while less is known about the regimes of unsteady laminar convection and turbulent convection occurring at high Rayleigh numbers. Only a few studies have been made of convection created by more complicated distributions of heat sources and sinks, as occur in many geophysical applications.

Restricting the attention to convection in a finite region, one finds that any fluid particle carries out a periodic or quasi-periodic motion between the top and the bottom, being heated at the lower portion and cooled at the upper portion of its orbit. It is suggested that insight into the mechanism of such a convection can be obtained by considering a very simple model: a one-dimensional fluid moving along a given closed loop and subjected to given heat sources and sinks along its path. The model can be materialized as follows: take a narrow tube of uniform cross-section and form it into a closed loop. The tube is filled with fluid that is kept well mixed over the cross-section (the characteristic time for the mixing should be small compared with the time required for the fluid to be advected an appreciable distance along the loop). If the fluid is turbulent the transverse mixing is usually strong enough but if the fluid is laminar one may have to introduce an artificial 'diffusor' in the tube. The tube walls are kept at a prescribed temperature that varies along the loop. The heat transfer to the fluid can be assumed proportional to the difference between the local temperature of the wall and the fluid.†

The fluid motion will be driven by the buoyancy force; it will also be retarded by a frictional force. One can with good approximation introduce a frictional force that is a function of the instantaneous flow rate; this is justified by the assumption that momentum like the heat is diffused over the cross-section in a time short compared with the 'advection time'.

The model described will have one solution of steady flow, whatever arrangements of heat sources and sinks is made. This can be proved as follows. The buoyancy force directed along the tube, integrated around the loop, is

$$\bar{B} = Ag\rho_0\alpha \oint T\,dz,$$

where A is the cross-sectional area, g the acceleration of gravity, ρ_0 a mean density, α the thermal expansion coefficient, T the temperature and z a vertical co-ordinate. In a steady state, this force is balanced by a tangential friction force $\bar{F}$, integrated along the tube. By assumption, $\bar{F}$ is a function of the flow rate q, $\bar{F}(q)$. $\bar{F}(q)$ obviously increases with q, for any normal friction law. A typical curve $\bar{F} = \bar{F}(q)$ is drawn in figure 1*a*. For any value of q one may further compute the temperature distribution in the fluid using the assumed heating law and from this obtain $\bar{B}(q)$. The relevant equation is $(q/A)(\partial T/\partial s) = k(T_0(s) - T)$, where s is a co-ordinate along the tube, $T_0(s)$ the wall temperature and k a constant. It is easy to show, and it is physically obvious, that when q becomes large the variations in T along the tube become small. Further, when $q = 0$ one has $T = T_0$ and $\bar{B}$ is non-zero (except in the case when $\oint T_0 dz = 0$, but then zero motion is one solution). Obviously the two curves must cut at least in one point; this represents a possible steady solution. The present proof assumes that the Boussinesq approximation is valid, but it seems possible to generalize this.

† One may feel that the fluid should take on exactly the wall temperature if the transverse mixing is instantaneous. However, in any experiment one finds a thermal resistance in a thin boundary layer at the wall, or in the wall itself. The heat flux can be assumed proportional to the temperature drop over this region. The inner core of the fluid can still be considered well mixed.

Looking on the steady (non-zero) convection it seems likely, at the first glance, that it is stable. If the flow rate should increase above its balanced value, the buoyancy forces would decrease and friction would increase, thereby counteracting the change. In the case where zero motion is a steady solution, one can obviously get instability. It is easy to show that, in this case, instability occurs

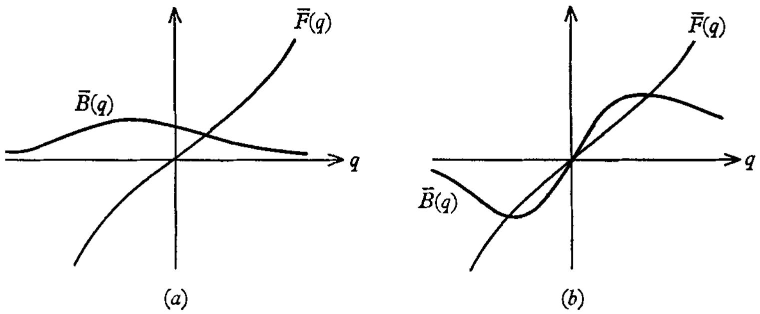

FIGURE 1. Variation of total friction $\bar{F}$ and total buoyancy $\bar{B}$ with flow rate; (*a*) no equilibrium, (*b*) one unstable equilibrium. (Schematic picture.)

when $\bar{B}'(q)$ is larger than $\bar{F}'(q)$, at $q = 0$. There exists, however, another steady solution (in fact, two solutions as seen from figure 1 *b*). It looks, therefore, as if the system would always end up in a stable situation. Some observations of a laboratory tube model suggested, however, that all steady solutions may eventually become unstable when the buoyancy force is made large. To test this idea some numerical experiments were run on the computer General Electric 225 at the Woods Hole Oceanographic Institution. The model studied was the simplest possible: a loop with two long vertical branches and a point heat source applied at the lower end, a point heat sink at the upper end. The computations demonstrated that the steady motion becomes unstable, and growing pulsative motions start when a non-dimensional gravity parameter is large enough, for a certain range of a frictional parameter. The instability was later demonstrated analytically; it was found to be associated with infinitesimal perturbations. The explanation in physical terms could also be given.

Instabilities of pulsative type have been reported for the Bénard convection at Rayleigh numbers around 10^6 by Thomas Rossby and for convection between rotating cylinders by Howard R. Snyder (private communications). In studies of theoretical models of the oceanic circulation driven by a non-uniform horizontal heating, there have also been reports on pulsative motions (Kirk Bryan, private communication), but these cases seem more complex and there is probably some more mechanism of instability involved. A further theoretical study of the pulsative instabilities for a real two- or three-dimensional fluid model would certainly be of interest and clarify the phenomenon further.

2. Derivation of the Model Equations

Consider the fluid in a portion of a tube with uniform cross-section area A and length L (figure 2). The fluid is driven by the pressure difference between the endpoints and by a buoyancy force, and is retarded by a frictional force. The following assumptions are made.

(i) The Boussinesq approximation is valid.

(ii) The tangential friction force on the fluid is proportional to the instantaneous flow rate q.

(iii) The temperature of the fluid is uniform over each cross-section.

(iv) The heat flux between the tube and the fluid is proportional to the difference between the wall temperature T_0 and the fluid temperature T. T_0 is prescribed along the tube.

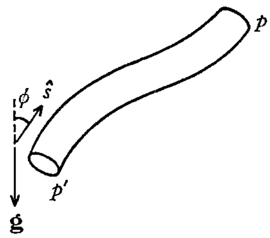

FIGURE 2. Tube segment.

The equations of motion for the fluid are

$$\rho_0 d\mathbf{v}/dt = -\nabla p + \rho \mathbf{g} + \mathbf{F}, \tag{1}$$

$$\nabla . \mathbf{v} = 0, \tag{2}$$

where $\mathbf{v}$ is the fluid velocity, ρ_0 a standard density, ρ the actual density, $\mathbf{g}$ the acceleration of gravity, and $\mathbf{F}$ the friction force. The density and temperature have a relation of the form

$$\rho = -\rho_0 \alpha T + \text{constant}. \tag{3}$$

Taking the tangential component of (1), one has

$$\rho_0 (d\mathbf{v}/dt)_s = -(\partial p/\partial s) - \rho g \cos\phi + F_s,$$

and, integrating over the fluid volume,

$$\rho_0 \frac{\partial}{\partial t}\iint v_s dA\, ds = -Al(p'-p) - g\iint \rho\cos\phi\, dA\, ds + \iint F_s dA\, ds,$$

where $p'-p$ is the pressure difference between the end-points, and ϕ is the angle between the tube and the vertical.

We introduce the flow rate $q = \int v_s dA$, which is constant along the tube in view of (2), and replace ρ by T using (3). Under the assumptions (ii) and (iii) the above equation can be written

$$\rho_0 L\dot{q} = -Al(p'-p) + Ag\rho_0\alpha\int T\, dz + \text{const.}\int dz - \rho_0 LRq, \tag{4}$$

where R is a frictional coefficient. Use has been made of the relation $\cos\phi\, ds = dz$ where dz is a vertical increment.

A second equation is given by the heat flux condition

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{q}{A}\frac{\partial T}{\partial s} = k\{T_0(s) - T\}. \tag{5}$$

We note that R and k both have the dimension of time $^{-1}$; these times characterize the viscous and thermal dissipation, respectively.

For a closed tube, equation (4) takes on the simple form

$$\dot{q} = \frac{A}{L} g\alpha \oint T\,dz - Rq \tag{4a}$$

since $p' = p$ and $\oint dz = 0$.

In the following we are going to restrict ourselves to the simple model shown in figure 3. The tube consists of two vertical branches of lengths $\frac{1}{2}L$ with short connexions at the top and the bottom. The tubes are assumed insulated ($k = 0$) everywhere except at the top and bottom, where over short distances s the temperature of the wall is kept at $-\Delta T$ and $+\Delta T$, respectively. We consider actually the limiting case when $\Delta s \to 0$ while $k \to \infty$, in such a way that the heat flux remains finite (point source and point sink).

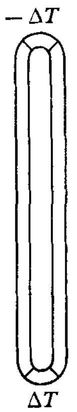

FIGURE 3. The simplified model.

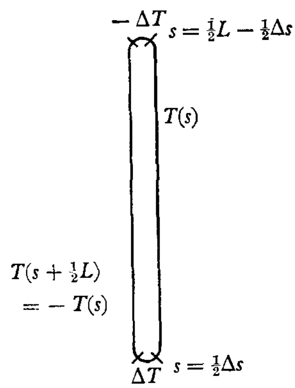

FIGURE 4. Co-ordinate along the tube, and antisymmetry condition.

The tube is symmetric with respect to the vertical. It is easy to see that by starting from a temperature distribution that is antisymmetric with respect to the centre of the loop, such a state of antisymmetry is always retained, see figure 4. One can, in fact, prove (not shown here) that starting from any temperature distribution the antisymmetric state is approached asymptotically at least in the case when q has one sign, so that each part of the fluid passes repeatedly through the heat source and heat sink.

It will, therefore, be natural to restrict the attention to this case. In the equations we let s vary in the interval from 0 to $\frac{1}{2}L$, and use the antisymmetry to include contributions from the interval $\frac{1}{2}L$ to L. The equation of motion is

$$\dot{q} = \frac{2Ag\alpha}{L} \int_{s=0}^{\frac{1}{2}L} T\,ds - Rq. \tag{6}$$

The temperature of the fluid coming out from the heat source or heat sink is determined by the temperature of the ingoing fluid and the flow rate only. It is easily seen that, if a fluid particle passes the heat source during the short time Δt, the difference in temperature between the outgoing and incoming fluid is, to the first order,

$$T_{\text{out}} - T_{\text{in}} = (\Delta T - T_{\text{in}})(1 - e^{-k\Delta t}) = (\Delta T - T_{\text{in}})(1 - e^{-kA\Delta s/|q|}). \tag{7}$$

For small flow rates $T_{\text{out}} = \Delta T$; for large flow rates $T_{\text{out}} = T_{\text{in}}$. For the heat sink the same formula holds with ΔT replaced by $-\Delta T$.

For the computation in the range $0 < s < \frac{1}{2}L$ we need to know T_{in} at the heat source, when $q > 0$, and at the heat sink, when $q < 0$. By the antisymmetry we have in the first case $T_{\text{in}} = -T_{s=\frac{1}{2}(L-\Delta s)}$, and in the second case $T_{\text{in}} = -T_{s=\frac{1}{2}\Delta s}$. In the limit when $\Delta s \to 0$ we denote these values simply $-T_{s=\frac{1}{2}L}$ and $-T_{s=0}$, respectively.

Finally, the equations are non-dimensionalized by the transformation

$$s \to \frac{L}{2}.s, \quad t \to \frac{L}{2k\Delta s}.t, \quad q \to kA\Delta s.q, \quad T \to \Delta T.T, \tag{8}$$

and take then the form

$$\dot{q} + \epsilon q = a \int_0^1 T\,ds, \tag{9}$$

$$(\partial T/\partial t) + q(\partial T/\partial s) = 0, \tag{10}$$

$$T_{s=0} + T_{s=1} = (1 + T_{s=1})(1 - e^{-1/q}) \quad \text{for} \quad q > 0, \tag{11a}$$

$$T_{s=0} + T_{s=1} = (-1 + T_{s=0})(1 - e^{1/q}) \quad \text{for} \quad q < 0. \tag{11b}$$

The parameters of the problem are

$$a = g\alpha \Delta T L / 2(k\Delta s)^2, \quad \epsilon = RL/2k\Delta s. \tag{12}$$

3. Steady motion

We assume that $q > 0$; hence the motion is upward in the branch $0 < s < 1$.

In steady state the temperature in this branch is uniform; its value is denoted by $\overline{T}$. The other branch has the temperature $-\overline{T}$. Denoting the flow rate $\overline{q}$, (9) and (11a) take on the form

$$\epsilon \overline{q} = a\overline{T}, \tag{13}$$

$$2\overline{T} = (1 + \overline{T})(1 - e^{-1/\overline{q}}). \tag{14}$$

Eliminating $\overline{T}$ one gets

$$\frac{2\overline{q}}{(a/\epsilon) + \overline{q}} = 1 - e^{-1/\overline{q}}. \tag{15}$$

It is easily seen that (15) has a single, positive solution. Values $\overline{q}$ and $\overline{T}$ are given in table 1 for some different values of a/ϵ.

Obviously the temperature becomes more uniform along the loop as the flow rate increases. For very small flow rates the temperature of the rising and sinking motion approaches the temperatures of the heat source and heat sink, respectively. The case of no motion is degenerate, because of the assumption of insulated branches. Any temperature distribution that satisfies the condition $\oint T\,dz = 0$ and that takes on the temperatures $+1$ and -1 at the heat source and heat sink will give an equilibrium solution. It is found that this equilibrium solution is unstable, at least when the initial temperatures lie between -1 and 1. Physically, this is seen by making a small displacement of the fluid from the equilibrium, say, in the positive direction. The fluid coming out from the heat source (or the heat sink) must necessarily have a higher (lower) temperature than the fluid that enters, and a positive buoyancy is created. When the branches are insulated this added buoyancy is conserved and the system will accelerate in the positive direction.

a/ϵ	$\bar{q}$	$\bar{T}$
0·100	0·100	1·000
0·500	0·417	0·834
1·000	0·648	0·648
2·000	0·958	0·479
10·000	2·218	0·222

TABLE 1. Values of non-dimensional flow rate and temperature in steady motion

4. Stability of steady motion

We consider small deviations from the steady state, putting

$$q = \bar{q}+q', \quad T = \bar{T}+T', \tag{16}$$

where q' is a function of t, and T' is a function of t and s. $\bar{q}$ and $\bar{T}$ are assumed positive.

The linearized forms of (9) and (10) are

$$\dot{q}' + \epsilon q' = a\int_0^1 T'\,ds, \tag{17}$$

$$(\partial T'/\partial t) + \bar{q}(\partial T'/\partial s) = 0, \tag{18}$$

while the boundary condition (11a) takes on the form

$$T'_{s=0} + mT'_{s=1} + nq' = 0, \tag{19}$$

where
$$m = e^{-1/\bar{q}} = \frac{1-\bar{T}}{1+\bar{T}}, \quad n = \frac{1+\bar{T}}{\bar{q}^2}\,e^{-1/\bar{q}} = \frac{1-\bar{T}}{\bar{q}^2}, \tag{20}$$

using the relation (14) between $\bar{q}$ and $\bar{T}$.

Introducing a time factor by putting

$$q' = \hat{q}\,e^{rt}, \quad T' = \hat{T}(s)\,e^{rt}, \tag{21}$$

the equations for $\hat{q}$ and $\hat{T}(s)$ are

$$(r+\epsilon)\,\hat{q} = a\int_0^1 \hat{T}\,ds, \tag{22}$$

$$r\hat{T} + \bar{q}(d\hat{T}/ds) = 0, \tag{23}$$

with the boundary condition

$$\hat{T}(0) + m\hat{T}(1) + n\hat{q} = 0. \tag{24}$$

The solutions for $\hat{T}$ and $\hat{q}$ are of the form

$$\left.\begin{aligned} \hat{T} &= C\exp(-rs/\bar{q}), \\ \hat{q} &= \frac{a}{r+\epsilon}\int_{s=0}^{1}\hat{T}\,ds = \frac{Ca\bar{q}}{r(r+\epsilon)}(1-e^{-r/\bar{q}}), \end{aligned}\right\} \tag{25}$$

and inserting these in the boundary condition (24) gives the characteristic equation

$$1+m\exp(-r/\bar{q})+\frac{na\bar{q}}{r(r+\epsilon)}\{1-\exp(-r/\bar{q})\} = 0. \tag{26}$$

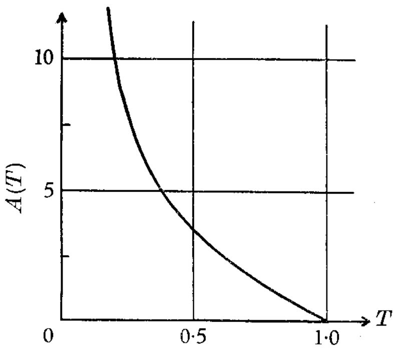

FIGURE 5. The function $A(T) = (1-T^{-2})\ln(1-T)/(1+T)$.

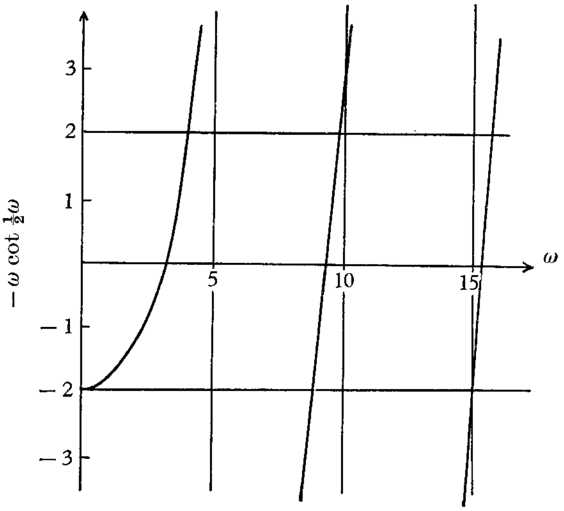

FIGURE 6. The function $-\omega\cot\frac{1}{2}\omega$.

For r real and positive, all the terms in the left-hand side of (26) are positive; hence no unstable solutions with exponential growth exist. There may still exist solutions that are unstable in an oscillatory way. The neutral oscillations are obtained by putting $r = i\omega$. In this case, after introducing the new parameters

$$\tilde{a} = na/\bar{q}, \quad \tilde{\epsilon} = \epsilon/\bar{q}, \quad \tilde{\omega} = \omega/\bar{q}, \tag{27}$$

the characteristic equation (26) can be written

$$\frac{\exp(i\tilde{\omega})+m}{\exp(i\tilde{\omega})-1}+\frac{\tilde{a}}{i\tilde{\omega}(i\tilde{\omega}+\tilde{\epsilon})} = 0. \tag{28}$$

Separating in real and imaginary parts, expressing $\tilde{a}$ in terms of $\bar{T}$ and $\tilde{\epsilon}$ only,† and inserting $m = (1-\bar{T})/(1+\bar{T})$, the following two equations result:

$$\tilde{\omega}^2+(\tilde{\epsilon}-\tfrac{1}{2}A)^2 = (\tfrac{1}{2}A)^2, \tag{29}$$

$$\tilde{\epsilon} = -(1/\bar{T})\tilde{\omega}\cot\tfrac{1}{2}\tilde{\omega}, \tag{30}$$

where

$$A = A(\bar{T}) = -\frac{1-\bar{T}^2}{\bar{T}^2}\ln\frac{1-\bar{T}}{1+\bar{T}}. \tag{31}$$

† Note the relations

$$\tilde{a} = \frac{\epsilon^3}{a}\frac{1-\bar{T}}{\bar{T}^3}, \quad \frac{\epsilon}{a} = -\bar{T}\ln\frac{1-\bar{T}}{1+\bar{T}} \quad \text{and} \quad \tilde{\epsilon} = \frac{\epsilon^2}{a\bar{T}}.$$

We note that $A > 0$, since $0 < \overline{T} < 1$. The behaviour of $A(\overline{T})$ is seen in figure 5. In an $(\tilde{\omega}, \tilde{\epsilon})$-plane (29) represents a circle with centre at $\frac{1}{2}A$ and diameter A. The curve represented by (30) is shown in figure 6 (for $\overline{T} = 1$). When $\overline{T}$ is small, which means large A, the two curves must obviously intersect, and two neutral

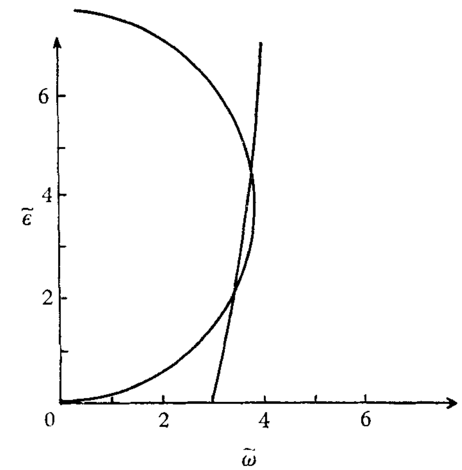

Figure 7. Construction of points on the neutral curve.

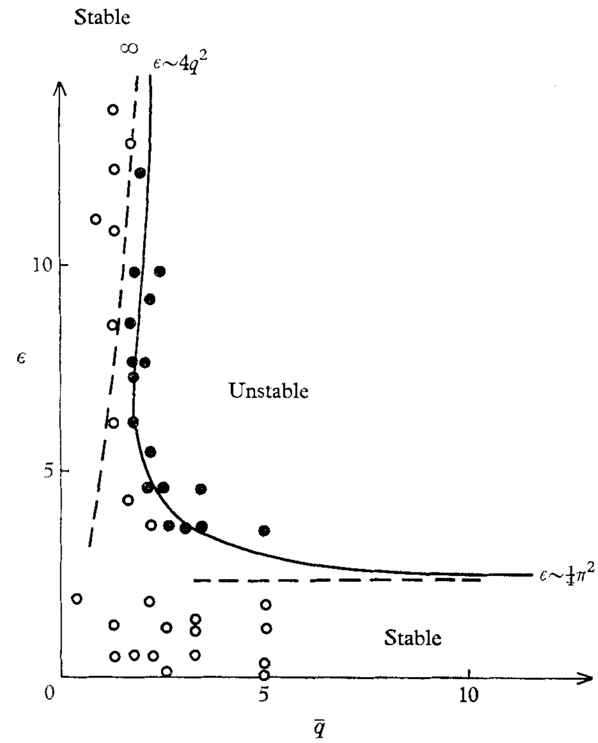

FIGURE 8. The neutral curve in a $(\bar{q}, \epsilon)$-plane, computed from the stability theory. The cases run in the numerical integrations are indicated by an open circle (stable) or a solid circle (unstable).

solutions exist (figure 7). The limiting case when the two curves first make contact occurs for $\bar{T} = 0{\cdot}265$. It is the largest $\bar{T}$ for which the flow can become unstable, and corresponds to a smallest possible flow rate $\bar{q} = 1{\cdot}84$. The angular frequency of the oscillation is $3{\cdot}60\bar{q}$. Since the angular frequency of a fluid particle is $\pi\bar{q}$, the thermal disturbance thus travels slightly faster than the fluid.

As the flow rate is increased above the critical value there is one upper and one lower value of ϵ for which neutral oscillations occur; in the range between these the solutions are amplified. The curve in the $(\bar{q}, \epsilon)$-plane representing neutral oscillations has been computed graphically, from the construction shown in figure 7. The result is shown in figure 8. In the asymptotic limits the upper and lower values of ϵ are $4\bar{q}^2$ and $\frac{1}{4}\pi^2$, respectively. This result is easily found, noticing that the two intersections between the circle and the $-\tilde{\omega}\cot\frac{1}{2}\tilde{\omega}$ curve approach $\tilde{\omega}_1 = 2\pi$, $\tilde{\epsilon}_1 = A$ and $\tilde{\omega}_2 = \pi$, $\tilde{\epsilon}_2 = \pi^2/A$ when the radius of the circle gets large, and that A itself has the asymptotic expression $4\bar{q}^2$.

The neutral curve constructed corresponds to the first mode, in which the temperature disturbance shows one wave along the loop. For larger $\bar{q}$ values one finds also higher modes with two, three, etc., waves along the loop. These come from intersections of the circle with the different branches of the $-\tilde{\omega}\cot\frac{1}{2}\tilde{\omega}$ curve. It is easily shown that the corresponding neutral curves all are enclosed by the curve for the first mode.

The present stability problem does not change qualitatively in the more general case where the frictional resistance is a non-linear function of q, as is expected in turbulent flows. ϵ is, however, now a function of $\bar{q}$, and this leads to a certain distortion in the previous neutral curve.

5. A numerical experiment

Studies of the non-linear oscillations were made by numerical integration of the equations (9), (10) and (11 *a, b*). These integrations were carried out on the computer General Electric 225 at the Woods Hole Oceanographic Institution. The flow rate q was directly obtained in curve form, by use of a machine plotter, while the temperature field was printed only at certain times by command of the operator. The computation over one cycle required about 1 sec without printing, and about 1 min with printing at every time step.

For the integration the loop was divided in 16, or sometimes 32, equal segments, each having a uniform temperature. Starting with zero temperature everywhere except in the first segment next to the heat source, where the temperature was assumed to be 1, the acceleration was computed by use of (9). The time required to advect the fluid one segment forward along the loop was then found, keeping the acceleration constant, and the temperature pattern was advanced one step. The temperature of the fluid segment passing the heat source could be estimated from a knowledge of the incoming fluid temperature and the mean flow rate over the step, using (11 *a, b*).

The results of four different runs are shown in figure 9 *a–d*. In the first case the system is damped and the approach to the steady state is aperiodic. In the second case the system is still stable, but the approach to the steady state is oscillatory.

In the third case we are very close to a neutral oscillation. A slight change in the parameter values can change the solution to one that slowly amplifies. The amplitude will be limited by non-linear effects and a steady pulsation develops. The temperature field over one cycle in a pulsation is shown in figure 10, while the variation of the flow rate and the total buoyancy $\oint T\,dz$ is shown in figure 11.

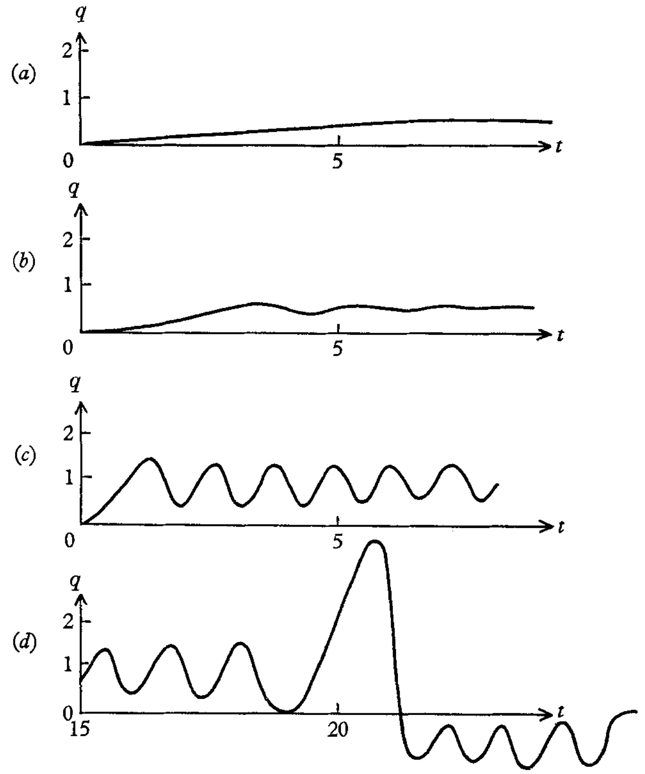

FIGURE 9. Result of numerical integration in four cases: (a) $a = 0{\cdot}4$, $\epsilon = 0{\cdot}2$; (b) $a = 2{\cdot}0$, $\epsilon = 1{\cdot}0$; (c) $a = 20{\cdot}0$, $\epsilon = 3{\cdot}0$; (d) $a = 40{\cdot}0$, $\epsilon = 6{\cdot}0$.

When the instability is made stronger a new phenomenon occurs that is exemplified in figure 9*d*. The non-linear effects cannot limit the pulsations but these grow until in one 'back-oscillation' the flow rate changes sign. The system then flips over and pulsations build up around a reversed mean flow, etc. One would perhaps expect to see a periodic behaviour develop, but in the numerical experiment the oscillations never repeated themselves, even during integration over several hundred cycles.

A similar 'ergodic' oscillation has been discussed in a report by Moore & Spiegel (1966). In their case the oscillator was even simpler, consisting of a single particle elastically restrained in an unstable surrounding fluid. The oscillations, which could be described by a third-order ordinary differential equation, became irregular because of the existence of a number of (three-dimensional) limit cycles that pass very close to each other at the equilibrium point. Even a small numerical error near this point could shift the system into another limit-cycle and in the course of time a large departure would be found. In the present case there is no indication of such a phenomenon. Numerical integrations using different numbers of segments have been carried out and compared, and it does not look as if numerical errors cause the irregularity. Rather it seems that the explanation lies in the large number of degrees of freedom of the system. With a continuous

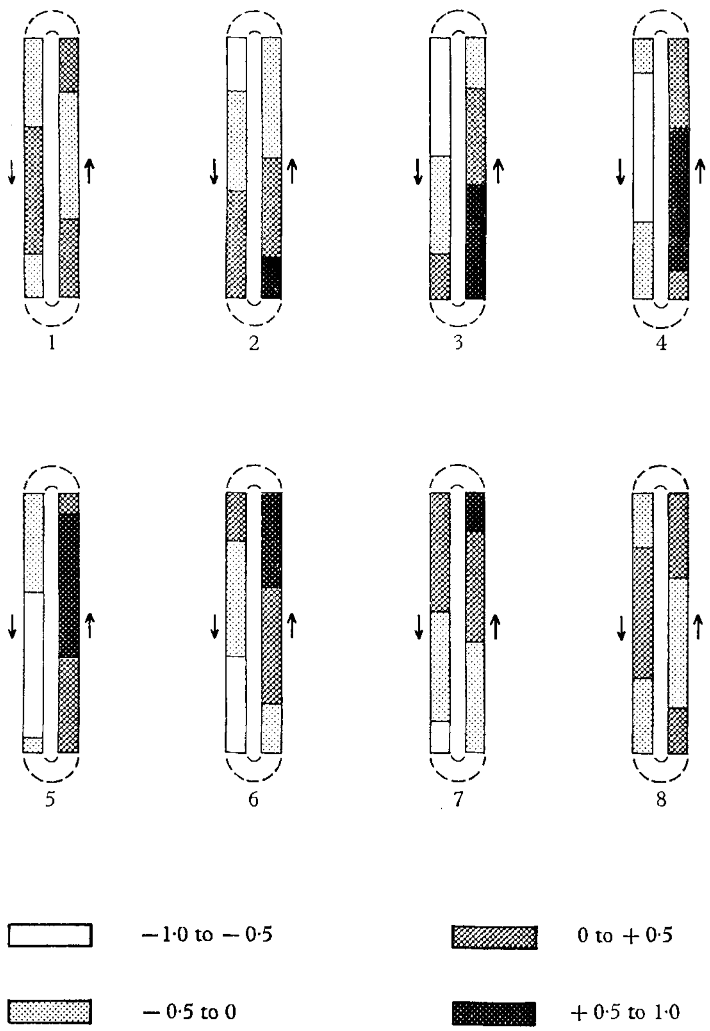

FIGURE 10. Variation in the temperature field over one cycle in an almost neutral oscillation ($a = 20{\cdot}0$, $\epsilon = 3{\cdot}0$).

distribution of temperature along the loop the number of degrees of freedom is, of course, infinite. In the actual integrations the number is finite, proportional to division in segments, but this number is so large that any periodic or quasi-periodic behaviour may come out only after a very long integration.

Using the numerical results it is possible to test the prediction of the linear stability theory. In figure 8 points are shown representing the numerical cases in a $(\bar{q}, \epsilon)$-plane. One sees that the transition from stable to unstable solutions occurs near the theoretical neutral curve, but that a certain increase of the unstable region is present. This may be an effect of the difference approximation and with a more accurate numerical method the agreement could probably be improved.

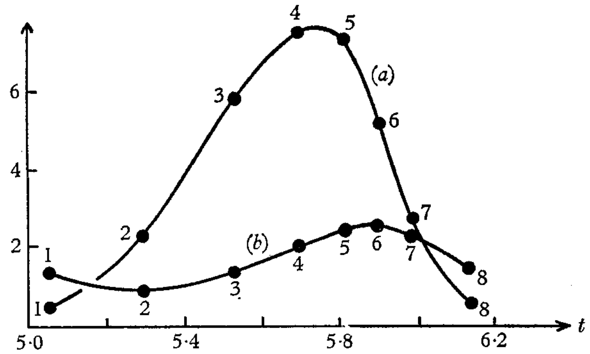

FIGURE 11. Variation in the total buoyancy (a) and flow rate (b) in the same case as represented in figure 10.

6. Physical explanation of the instability

In a steady motion one finds viscous and thermal dissipation that seem to oppose any change in the flow rate. Any increase in flow rate, for example, would cause an increase in friction and a decrease in total buoyancy, because the heating/unit length of fluid column is diminished. One may therefore think that a steady motion would always be stable. However, these two restraining effects may not be in phase and an overshooting can then occur, eventually producing growing oscillations.

In the present case the mechanism of instability is made clear by figures 10 and 11. One sees in figure 10 a positive thermal anomaly, or 'warm pocket', that emanates from the heat source and follows the fluid up along the right branch (stages 2–5). It then passes the heat sink (stages 6–7) and can still be traced in the fluid running down the left branch. Passing the heat source it again appears amplified. In the opposite position one sees similarly a 'cold pocket' that is regenerated each time it passes the heat sink. The variation of buoyancy and flow rate shown in figure 11 explains this regeneration. As the 'warm pocket' comes out from the heat source, positive buoyancy is built up and the flow accelerates. When the pocket passes the heat sink the flow rate is maximum (stage 6). Thus the effect of the heat sink is minimized. Half a cycle later when the 'warm pocket' passes the heat source, the flow rate is minimum and the heating can regenerate the anomaly effectively. A similar reasoning holds for the 'cold pocket'. One may see the mechanism most simply by considering the fluid as a pendulum, with its mass centre towards the 'cold pocket'. The heat source and sink will be most effective when the pendulum is in its upper slower motion, and less effective when the pendulum is in its lower more rapid motion.

One also wants to understand why, for a given mean flow rate $\bar{q}$, oscillations occur only for a certain range of the friction parameter ϵ, excluding the very small and very large values. We note that, for a fixed $\bar{q}$, the limits of small and large ϵ represent also the limits of small and large values, respectively, of the buoyancy parameter a ($\bar{q}$ is a function of ϵ/a alone).

In the limit of large ϵ one finds that the acceleration term in (9) is small, and buoyancy and friction balance. The lag between the buoyancy and flow shown in figure 11 is no longer present. When the warmest fluid reaches the cold source and the buoyancy decreases (stages 5–6) the velocity immediately drops. The effectiveness of the instability mechanism discussed earlier is therefore diminished. In the limit of a small ϵ, and thus a small a, (9) requires, on the other hand, that $\bar{q}$ is almost constant. Thus the fluid has a large inertia. The flow can build up slowly under the influence of a steady buoyancy force. However, it will not react much to buoyancy variations in one cycle, and again the effectiveness of the instability mechanism is diminished.

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REFERENCES

Keller, J. 1966 *J. Fluid Mech.* **26**, 599–606.

Moore, D. & Spiegel, E. A. 1966 A thermally excited non-linear oscillator. *Astrophys. J.* **143**, 871–887.